PARAMETERS ASSOCIATED WITH BIVARIATE BERNSTEIN-SZEGO MEASURES ON THE BI-CIRCLE

JEFFREY S. GERONIMO AND PHILIP BENGE*

ABSTRACT. We consider measures supported on the bi-circle and review the recurrence relations satisfied by the orthogonal polynomials associated with these measures constructed using the lexicographical or reverse lexicographical ordering. New relations are derived among these recurrence coefficients. We extend the results of [8] on a parameterization for Bernstein-Szego measures supported on the bi-circle.

1. INTRODUCTION

In this paper we continue the investigation begun in [8] on the orthogonal polynomials associated with measures supported on the bi-circle. In more than one variable an important consideration is which ordering to use. The usual ordering is the one suggested by Jackson [10] which is the the total degree ordering. This is natural since the addition of any new polynomials does not alter the previous orthogonal polynomials already constructed. However in their solution of the two-variable Fejer-Reisz problem Geronimo and Woerdeman [7] were led to consider polynomials obtained using the lexicographical or reverse lexicographical ordering (for an alternative viewpoint see Knese [11]). Orthogonal polynomials obtained using these orderings were first studied by Delsarte et al [2] who used them to solve the half-plane least squares problem [3]. Important in their work and later emphasized in [8] is the fact that in these orderings the moment matrices have a doubly Toeplitz structure. This allows a connection between the polynomials obtained using the above orderings and matrix orthogonal polynomials on the unit circle [2] (see also [8]). The various convergence properties of these polynomials in a strip as well as their connection to generalized Schur representations and Adamjan, Arov, and Krein theory were developed in [5], and [6].

Given a positive Borel probability measure σ supported on the unit circle with an infinite number of points of increase let $\{\phi_n\}_{n\geq 0}$ be the sequence of polynomials of exact degree n in $z = e^{i\theta}$ with positive leading coefficient having the property,

$$\int_{\mathbb{T}} \phi_j(e^{i\theta})\overline{\phi_k(e^{i\theta})}d\sigma(\theta) = \delta_{j,k}$$

where $\delta_{j,k}$ is the Kronecker delta. These polynomials are known to satisfy the recurrence formula [9], [12], [13],

$$\phi_n(z) = a_n(z\phi_{n-1}(z) - \alpha_n \overleftarrow{\phi}_{n-1}(z)),\ n \geq 1$$

where $\overleftarrow{\phi}_n(z) = z^n\bar{\phi}_n(1/z)$ is called the reverse polynomial. The α_n are called the recurrence coefficients and $a_n = \frac{k_{n-1}}{k_n}$ where k_n is the leading coefficient of ϕ_n.

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From the orthogonality properties of ϕ_n and ϕ_{n-1} it is not difficult to obtain the relation

$$1 = a_n^2(1 - |\alpha_n|^2).$$

Thus $|\alpha_n| < 1$ and given α_n, a_n can be computed. The recurrence coefficients play an important role in theory of orthogonal polynomials on the unit circle as can be seen by Verblunsky's Theorem [12].

Theorem 1.1. *Let σ be a Borel probability supported on the unit circle with an infinite support, then associated with σ is a unique sequence of recurrence coefficients $\{\alpha_n\}_{n=1}^{\infty}$ with $|\alpha_n| < 1$. This correspondence is one-to-one.*

A useful characterization theorem is the following,

Theorem 1.2. *Let σ be a Borel measure supported on the unit circle. Then σ is absolutely continuous with respect to Lebesgue measure with density $\frac{1}{|p_n(z)|^2}$ where $p_n(z)$ is a polynomial of exact degree n in z with $\overleftarrow{p}_n(z)$ nonzero for $|z| \le 1$ if and only if $\alpha_i = 0$, for $i > n$.*

Measures of the form given in the above Theorem have come to be called Bernstein-Szego measures [12].

In [8] a parameterization of the two variable trigonometric moment problem was introduced in an attempt to be able to find extensions of the above two Theorems to the two variable case. Here we continue the study of the algebraic properties of this problem. In section 2 we call together the results needed. In particular beginning with Borel measures supported on the bi-circle whose moment matrices are positive we construct orthogonal polynomials using the lexicographical or reverse lexicographical ordering. Then recurrence relations satisfied by these polynomials are displayed and some properties of the recurrence coefficients are noted. Here the parameterization discussed above is introduced and a two variable analog of Verblunsky's Theorem is presented. In section 3 we develop new equations between the recurrence coefficients that shed light on how these coefficients are related to each other. Some of these recurrence relations are used in section 4 to develop an algorithm different from that given in [8] which allows us to make more precise the construction of the parameters left undetermined in Theorem 7.9 of [8].

2. Preliminaries

In this section we collect some results that will be used later. As noted above we will use the lexicographical ordering which is defined by

$$(k,\ell) <_{\mathrm{lex}} (k_1,\ell_1) \Leftrightarrow k < k_1 \text{ or } (k = k_1 \text{ and } \ell < \ell_1),$$

and the reverse lexicographical ordering, defined by

$$(k,\ell) <_{\mathrm{revlex}} (k_1,\ell_1) \Leftrightarrow (\ell,k) <_{\mathrm{lex}} (\ell_1,k_1).$$

Both of these orderings are linear orders, and in addition they satisfy

$$(k,\ell) < (m,n) \Rightarrow (k+p,\ell+q) < (m+p,n+q).$$

In such a case, one may associate a half-space with the ordering which is defined by $\{(k,l) \;:\; (0,0) < (k,l)\}$. In the case of the lexicographical ordering we shall denote the associated half-space by H and refer to it as *the standard half-space*. In the case of the reverse lexicographical ordering we shall denote the associated half-space by

$\tilde{H}$. Let σ be a positive Borel measure support on the bi-circle $z = e^{i\theta}$, $w = e^{i\phi}$ with Fourier coefficients,

$$c_{k,j} = \int_{\mathbb{T}} e^{-ik\theta} e^{-ij\phi} d\sigma(\theta, \phi).$$

We now form the $(n+1)(m+1) \times (n+1)(m+1)$ moment matrix $C_{n,m}$ using the lexicographical ordering. As noted in the introduction it has a special block Toeplitz form

$$C_{n,m} = \begin{bmatrix} C_0 & C_{-1} & \cdots & C_{-n} \\ C_1 & C_0 & \cdots & C_{-n+1} \\ \vdots & & \ddots & \vdots \\ C_n & C_{n-1} & \cdots & C_0 \end{bmatrix}, \tag{2.1}$$

where each C_i is an $(m+1) \times (m+1)$ Toeplitz matrix as follows:

$$C_i = \begin{bmatrix} c_{i,0} & c_{i,-1} & \cdots & c_{i,-m} \\ \vdots & & \ddots & \vdots \\ c_{i,m} & & \cdots & c_{i,0} \end{bmatrix}, \qquad i = -n, \dots, n. \tag{2.2}$$

Thus $C_{n,m}$ has a doubly Toeplitz structure. If the reverse lexicographical ordering is used in place of the lexicographical ordering, we obtain another moment matrix $\tilde{C}_{n,m}$ where the roles of n and m are interchanged. Throughout the rest of the paper we will assume that $C_{n,m}$ is positive definite for all $0 \le n,\ m$.

We now compute orthogonal polynomials associated with σ. We begin by ordering the monomials $z^i w^j,\ 0 \le i \le n,\ 0 \le j \le m$ lexicographically then performing the Gram–Schmidt procedure using this ordering. Define the orthonormal polynomials $\phi^l_{n,m}(z, w),\ 0 \le n,\ 0 \le m,\ 0 \le l \le m$, by the equations

(2.3)

$$\int_{\mathbb{T}} \phi^l_{n,m} z^{-i} w^{-j} d\sigma = 0, \quad 0 \le i < n \text{ and } 0 \le j \le m \quad \text{ or } i = n \text{ and } 0 \le j < l,$$

$$\int_{\mathbb{T}} \phi^l_{n,m} \overline{\phi^l_{n,m}} = 1,$$

and

$$\phi^l_{n,m}(z, w) = k^{n,l}_{n,m,l} z^n w^l + \sum_{(i,j) <_{\text{lex}} (n,l)} k^{i,j}_{n,m,l} z^i w^j. \tag{2.4}$$

With the convention $k^{n,l}_{n,m,l} > 0$, the above equations uniquely specify $\phi^l_{n,m}$. Polynomials orthonormal with respect to σ but using the reverse lexicographical ordering will be denoted by $\tilde{\phi}^l_{n,m}$. They are uniquely determined by the above relations with the roles of n and m interchanged.

Set

$$\Phi_{n,m} = \begin{bmatrix} \phi^m_{n,m} \\ \phi^{m-1}_{n,m} \\ \vdots \\ \phi^0_{n,m} \end{bmatrix} = K_{n,m} \begin{bmatrix} z^n w^m \\ z^n w^{m-1} \\ \vdots \\ 1 \end{bmatrix}, \tag{2.5}$$

where the $(m+1)\times(n+1)(m+1)$ matrix $K_{n,m}$ is given by

$$K_{n,m} = \begin{bmatrix} k^{n,m}_{n,m,m} & k^{n,m-1}_{n,m,m} & \cdots & \cdots & \cdots & k^{0,0}_{n,m,m} \\ 0 & k^{n,m-1}_{n,m,m-1} & \cdots & \cdots & \cdots & k^{0,0}_{n,m,m-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & k^{n,0}_{n,m,0} & k^{n-1,m}_{n,m,0} & \cdots & k^{0,0}_{n,m,0} \end{bmatrix}. \tag{2.6}$$

As indicated above denote

$$\tilde\Phi_{n,m} = \begin{bmatrix} \tilde\phi^n_{n,m} \\ \tilde\phi^{n-1}_{n,m} \\ \vdots \\ \tilde\phi^0_{n,m} \end{bmatrix} = \tilde K_{n,m} \begin{bmatrix} w^m z^n \\ w^m z^{n-1} \\ \vdots \\ 1 \end{bmatrix}, \tag{2.7}$$

where the $(n+1)\times(n+1)(m+1)$ matrix $\tilde K_{n,m}$ is given similarly to (2.6) with the roles of n and m interchanged. For the bivariate polynomials $\phi^l_{n,m}(z,w)$ above we define the reverse polynomials $\overleftarrow{\phi}^l_{n,m}(z,w)$ by the relation

$$\overleftarrow{\phi}^l_{n,m}(z,w) = z^n w^m \bar\phi^l_{n,m}(1/z, 1/w). \tag{2.8}$$

With this definition $\overleftarrow{\phi}^l_{n,m}(z,w)$ is again a polynomial in z and w, and furthermore

$$\overleftarrow{\Phi}_{n,m}(z,w) := \begin{bmatrix} \overleftarrow{\phi}^m_{n,m} \\ \overleftarrow{\phi}^{m-1}_{n,m} \\ \vdots \\ \overleftarrow{\phi}^0_{n,m} \end{bmatrix}^T. \tag{2.9}$$

An analogous procedure is used to define $\overleftarrow{\tilde\phi}^l_{n,m}$ and $\tilde\Phi_{n,m}$.

To find recurrence formulas for the vector polynomials $\Phi_{n,m}$, we introduce the notation for every vector valued polynomials X and Y,

$$\langle X, Y\rangle = \int_{\mathbb{T}} X(z,w) Y^\dagger(z,w) d\sigma, \quad |z| = 1 = |w| \tag{2.10}$$

The following recurrence formulas which follow from the orthogonality relations satisfied by $\Phi_{n,m}$ and $\tilde\Phi_{n,m}$ were proved in [8].

Theorem 2.1. *Given* $\{\Phi_{n,m}\}$ *and* $\{\tilde\Phi_{n,m}\}$, $0<n$, $0<m$, *the following recurrence formulas hold:*

$$A_{n,m}\Phi_{n,m} = z\Phi_{n-1,m} - \hat E_{n,m}\overleftarrow{\tilde\Phi}^T_{n-1,m}, \tag{2.11}$$

$$\Phi_{n,m} + A^\dagger_{n,m}\hat E_{n,m}(A^T_{n,m})^{-1}\overleftarrow{\tilde\Phi}^T_{n,m} = A^\dagger_{n,m} z\Phi_{n-1,m}, \tag{2.12}$$

$$\Gamma_{n,m}\Phi_{n,m} = \Phi_{n,m-1} - \mathcal{K}_{n,m}\tilde\Phi_{n-1,m}, \tag{2.13}$$

$$\Gamma^1_{n,m}\Phi_{n,m} = w\Phi_{n,m-1} - \mathcal{K}^1_{n,m}\overleftarrow{\tilde\Phi}^T_{n-1,m}, \tag{2.14}$$

$$\Phi_{n,m} = I_{n,m}\tilde\Phi_{n,m} + \Gamma^\dagger_{n,m}\Phi_{n,m-1}, \tag{2.15}$$

$$\overleftarrow{\Phi}^T_{n,m} = I^1_{n,m}\tilde\Phi_{n,m} + (\Gamma^1_{n,m})^T\overleftarrow{\Phi}^T_{n,m-1}, \tag{2.16}$$

where

$$\hat{E}_{n,m} = \langle z\Phi_{n-1,m}, \overleftarrow{\Phi}^T_{n-1,m}\rangle = \hat{E}^T_{n,m} \in M^{m+1,m+1}, \tag{2.17}$$

$$A_{n,m} = \langle z\Phi_{n-1,m}, \Phi_{n,m}\rangle \in M^{m+1,m+1}, \tag{2.18}$$

$$\mathcal{K}_{n,m} = \langle \Phi_{n,m-1}, \tilde{\Phi}_{n-1,m}\rangle \in M^{m,n}, \tag{2.19}$$

$$\Gamma_{n,m} = \langle \Phi_{n,m-1}, \Phi_{n,m}\rangle \in M^{m,m+1}, \tag{2.20}$$

$$\mathcal{K}^1_{n,m} = \langle w\Phi_{n,m-1}, \overleftarrow{\tilde{\Phi}}^T_{n-1,m}\rangle \in M^{m,n}, \tag{2.21}$$

$$\Gamma^1_{n,m} = \langle w\Phi_{n,m-1}, \Phi_{n,m}\rangle \in M^{m,m+1}, \tag{2.22}$$

$$I_{n,m} = \langle \Phi_{n,m}, \tilde{\Phi}_{n,m}\rangle \in M^{m+1,n+1}, \tag{2.23}$$

$$I^1_{n,m} = \langle \overleftarrow{\Phi}^T_{n,m}, \tilde{\Phi}_{n,m}\rangle \in M^{m+1,n+1}. \tag{2.24}$$

Here $M^{i,j}$ denotes the set of $i\times j$ matrices with complex entries. Equation (2.11) was first found by Delsarte et. al. [2].

Remark 2.2. Formulas similar to (2.11)–(2.16) hold for $\tilde{\Phi}_{n,m}$ and will be denoted by ($\tilde{}$2.11)–($\tilde{}$2.16). Throughout the rest of the paper we use the same notation to denote the tilde analogues of existing formulas stated for $\Phi_{n,m}$.

Examination of equation (2.13) shows that the (i,j) entries of $\Gamma_{n,m}$ are zero for $i \geq j$ with the $(i,i+1)$ entries positive. Likewise equation (2.14) implies that entries (i,j) of $\Gamma^1_{n,m}$ are zero for $i > j$ with the (i,i) entries positive.

From the definitions of $\mathcal{K}$, $\mathcal{K}^1$, I, I^1 and their tilde analogs it is not difficult to see the following relations,

$$\tilde{\mathcal{K}}_{n,m} = \mathcal{K}^\dagger_{n,m},\ \tilde{I}_{n,m} = I^\dagger_{n,m}, \tag{2.25}$$

$$\tilde{I}^1_{n,m} = (I^1_{n,m})^T,\ \tilde{\mathcal{K}}^1_{n,m} = (\mathcal{K}^1_{n,m})^T. \tag{2.26}$$

Also the recurrence relations yield,

$$A_{n,m}A^\dagger_{n,m} = I_m - \hat{E}_{n,m}\hat{E}^\dagger_{n,m}, \tag{2.27}$$

$$\Gamma_{n,m}\Gamma^\dagger_{n,m} = I_m - \mathcal{K}_{n,m}\mathcal{K}^\dagger_{n,m}, \tag{2.28}$$

$$\Gamma^1_{n,m}(\Gamma^1_{n,m})^\dagger = I_m - \mathcal{K}^1_{n,m}(\mathcal{K}^1_{n,m})^\dagger, \tag{2.29}$$

$$I_{n,m}I^\dagger_{n,m} + \Gamma^\dagger_{n,m}\Gamma_{n,m} = I_{m+1}, \tag{2.30}$$

$$I^1_{n,m}(I^1_{n,m})^\dagger + (\Gamma^1_{n,m})^\dagger\Gamma^1_{n,m} = I_{m+1}. \tag{2.31}$$

The definition of $\Phi_{n,m}$ implies that $A_{n,m}$ is an upper triangular matrix with positive diagonal entries. Thus it may be computed from $\hat{E}_{n,m}$ using a Cholesky decomposition of equation (2.27). A slightly more involved analysis [8] shows that $\Gamma_{n,m}$ may be computed using a Cholesky decomposition of (2.28).

The above recurrence formulas also give pointwise formulas for the recurrence coefficients. In order to obtain these formulas we define the $m\times m+1$ matrices U_m and U^1_m as

$$U_m = \begin{bmatrix}0, & I_m\end{bmatrix}, \tag{2.32}$$

and

$$U^1_m = \begin{bmatrix}I_m, & 0\end{bmatrix}, \tag{2.33}$$

where I_m is the $m\times m$ identity matrix. From equations (2.39) and (2.40) we write

$$\Phi_n^m(z) = \Phi_{n,n}^m z^n + \Phi_{n,n-1}^m z^{n-1} + \cdots,$$
$$\tilde{\Phi}_m^n(w) = \tilde{\Phi}_{m,m}^n w^m + \tilde{\Phi}_{m,m-1}^n w^{m-1} + \cdots, \tag{2.34}$$

then the following relations hold:

$$\Gamma_{n,m} = \Phi_{n,n}^{m-1} U_m (\Phi_{n,n}^m)^{-1}, \tag{2.35}$$

$$\Gamma_{n,m}^1 = \Phi_{n,n}^{m-1} U_m^1 (\Phi_{n,n}^m)^{-1}, \tag{2.36}$$

$$\mathcal{K}_{n,m} = -\Gamma_{n,m} I_{n,m} \tilde{F}_{n,m}, \tag{2.37}$$

$$\mathcal{K}_{n,m}^1 = -\Gamma_{n,m}^1 \bar{I}_{n,m}^1 \bar{\tilde{F}}_{n,m}^1, \tag{2.38}$$

where $\tilde{F}_{n,m} = \tilde{\Phi}_{m,m}^n U_n^T (\tilde{\Phi}_{m,m}^{n-1})^{-1}$, and $\tilde{F}_{n,m}^1 = \tilde{\Phi}_{m,m}^n (U_n^1)^T (\tilde{\Phi}_{m,m}^{n-1})^{-1}$.

Equations (2.11) and (2.12) are a consequence of the relation between $\Phi_{n,m}$ and the matrix orthogonal polynomials associated with the $(m+1)\times(m+1)$ matrix measure M_m given by

$$dM_m(\theta) = \int_{\phi=-\pi}^{\pi} \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix} d\mu(\theta,\phi) \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix}^{\dagger},$$

where $w = e^{i\phi}$. Given $E_{i,m}$ equation (2.11) allows the computation of $\Phi_{i,m}$ along the strip $0 \le i$, $0 \le j \le m$. More precisely if we write

$$\Phi_{n,m}(z,w) = \Phi_n^m(z)[w^m,\dots,1]^T, \tag{2.39}$$

then the Φ_i^m are a sequence of matrix polynomials of degree i in z satisfying

$$\int_{-\pi}^{\pi} \Phi_i^m(z) dM(\theta) (\Phi_j^m(z))^{\dagger} = I_{m+1}\delta_{i,j},$$

where I_{m+1} is the $(m+1)\times(m+1)$ identity matrix and $\delta_{i,j}$ is the Kronecker delta. Thus (2.11) and (2.12) follow the recurrence formulas satisfied by matrix polynomials orthogonal on the unit circle. Equation (2.5) implies that the coefficient of z^i in Φ_i^m, $\Phi_{i,i}^m$, is an $(m+1)\times(m+1)$ upper triangular matrix with positive diagonal entries. Similar statements hold for

$$\tilde{\Phi}_{n,m}(z,w) = \tilde{\Phi}_n^m(w)[z^n,\dots,1]^T. \tag{2.40}$$

In contrast given $\Phi_{i,j}$,and $\tilde{\Phi}_{i,j}$ for $(i,j) = (n-1,m)$ or $(n,m-1)$ equations (2.13) and (2.14) allow the computation of $\Phi_{n,m}$.

As noted in [8] there is a lot of redundancy in the coefficients of the above equations. If we have all the Fourier coefficients in the notched rectangle $\{(i,j), 0 \le i \le n, 0 \le j \le m\} \setminus (n,m)$ then the polynomials $\Phi_{n,m-1}$, $\tilde{\Phi}_{n-1m}$ can be computed. Notice that only two new Fourier coefficients are required to compute all the polynomial $\Phi_{n,m}$ whereas $\mathcal{K}_{n,m}$ and $\mathcal{K}_{n,m}^1$ are both $m\times n$ matrices. This led in [8] to the introduction of parameters $u_{i,j}$ $i \ge 0$, $u_{-i,-j} = \bar{u}_{i,j}$ such that

$$u_{-n,-m} = (\mathcal{K}_{n,m}^1)_{1,1} \ n>0,\ m>0, \tag{2.41}$$

and

$$\begin{aligned} u_{-n,m} &= (e_m^m)^T (\Phi_{n,n}^{m-1})^{-1} \mathcal{K}_{n,m} ((\tilde{\Phi}_{m,m}^{n-1})^{\dagger})^{-1} e_n^n \\ &= (e_m^m)^T \mathcal{K}_{n,m} e_n^n / (k_{n,m-1,0}^{n,0} \tilde{k}_{n-1,m,0}^{m,0}),\ n>0,\ m>0, \end{aligned} \tag{2.42}$$

where e_m^m is the m-dimensional vector with zeros in all its entries except the last, which is one, and $k_{n,m-1,0}^{n,0}$ and $\tilde{k}_{n-1,m,0}^{m,0}$ are the leading coefficient of $\phi_{n,m-1}^0$ and $\tilde{\phi}_{n-1,m}^0$ respectively. The last equality was obtained using the upper triangularity of $\Phi_{n,n}^{m-1}$ and $\tilde{\Phi}_{m,m}^{n-1}$, and equations (2.5) and (2.7). In terms of inner products the parameters can be written as

$$u_{-n,-m} = \int_{\mathbb{T}^2} \phi_{n,m-1}^{m-1}(z,w)\overline{\overleftarrow{\tilde{\phi}}_{n-1,m}^{n-1}(z,w)}d\sigma(\theta,\phi),\ z = e^{i\theta},\ w = e^{i\phi}, \tag{2.43}$$

and

$$u_{-n,m} = \int_{\mathbb{T}^2} \hat{\phi}_{n,m-1}^0(z,w)\overline{\hat{\tilde{\phi}}_{n-1,m}^0(z,w)}d\sigma(\theta,\phi),\ z = e^{i\theta},\ w = e^{i\phi}. \tag{2.44}$$

Here $\hat{\phi}_{n,m-1}^0$ and $\hat{\tilde{\phi}}_{n,m-1}^0$ have leading coefficient one. Since $\mathcal{K}_{n,m}$ and $\mathcal{K}_{n,m}^1$ are contractions the parameters must satisfy the constraints

$$|u_{n,m}| < 1$$

and

$$k_{n,m-1,0}^{n,0}\tilde{k}_{n-1,m,0}^{m,0}|u_{n,-m}| < 1.$$

With this the following Theorem was proved in [8],

Theorem 2.3. *Given parameters $u_{i,j} \in \mathbb{C}$, $0 \le i$, $u_{-i,j} = \bar{u}_{i,-j}$ construct*

- *scalars $\hat{E}_{i,0}$ and $\hat{\tilde{E}}_{0,j}$;*
- *matrices $\mathcal{K}_{i,j}$, $i > 0$, $j > 0$; and*
- *numbers $(e_1^j)^T H_{i,j}^3 e_1^j$, $i > 0$, $j > 0$.*

If

$$u_{0,0} > 0,\ |\hat{E}_{i,0}| < 1, |\hat{\tilde{E}}_{0,j}| < 1,\ ||\mathcal{K}_{i,j}|| < 1,\ \textit{and } (H_{i,j}^3)_{1,1} < 1, \tag{2.45}$$

then there exists a unique positive measure σ supported on the bi-circle such that

$$\int_{\mathbb{T}} \Phi_{i,m}\Phi_{j,m}^\dagger d\sigma = \delta_{i,j}I_{m+1}\ \textit{and}\ \int_{\mathbb{T}} \tilde{\Phi}_{n,i}\tilde{\Phi}_{n,j}^\dagger d\sigma = \delta_{i,j}I_{n+1}. \tag{2.46}$$

The conditions (2.45) *are also necessary.*

The numbers $(H_{i,j}^3)_{1,1}$ are given by equation (5.14) in [8] .

This Theorem is the two dimensional analog of Verblunsky's Theorem discussed in the introduction. A polynomial p is of degree (n,m) if

$$p(z,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{i,j}z^i w^j,$$

with $p_{n,m} \ne 0$.

When $d\sigma = \frac{1}{|p_{n,m}(z,w)|^2}$ where $p_{n,m}$ is a polynomial of degree (n,m) with $\overleftarrow{p}_{n,m}$ stable (i.e. $\overleftarrow{p}_{n,m} \ne 0$, $|z|, |w| \le 1$) then more can be said.

Theorem 2.4. *Let μ be a positive measure on the bicircle. Then μ is purely absolutely continuous with respect to the Lebesgue measure and $d\mu = \frac{d\theta d\phi}{4\pi^2|p_{n,m}|^2}$, where $p_{n,m}$ is a polynomial of degree (n,m) with $\overleftarrow{p}_{n,m}$ stable if and only if*

(a) $\mathcal{K}_{n,j} = 0$, $\hat{\tilde{E}}_{n-1,j+1} = 0$, *and* $u_{n,j+1} = 0$, $j \ge m$;
(b) $\mathcal{K}_{i,m} = 0$, $\hat{E}_{i,m-1} = 0$, *and* $u_{i,m} = 0$, $i > n$;
(c) $u_{i,j} = 0, |i| > n,\ j > m$.

It was actually shown that in this case $u_{-n,j},\ u_{-i,m},\ u_{n-1,j+1}$ and $u_{i,m-1}$ are equal to zero for $j \geq m,\ i > n$. How to compute the remaining parameters and how they were related to $u_{i,j}$, where $|i| \leq n$ and $j \leq m$, was not indicated and it the subject of the remaining sections.

3. Relations for $\hat{E}_{n,m}$, $\mathcal{K}_{n,m}$ and $\mathcal{K}^1_{n,m}$

In order to prove Theorem 2.3 it was necessary to show that most of the entries in $\mathcal{K}_{n,m}$ and $\mathcal{K}^1_{n,m}$ could be computed knowing the recurrence coefficients on the $(n-1, m)$ and $(n, m-1)$ levels. These relations will be augmented by the following new relations which will be used later to compute coefficients on lower levels from those on higher levels.

Lemma 3.1. *For $n > 0$ and $m \geq 0$,*

$$\hat{E}_{n,m} = \Gamma_{n-1,m+1}\hat{E}_{n,m+1}(\Gamma^1_{n-1,m+1})^T + \mathcal{K}_{n-1,m+1}(\mathcal{K}^1_{n-1,m+1})^T. \tag{3.1}$$

Also for $n \geq 0$ and $m > 0$,

$$\hat{\tilde{E}}_{n,m} = \tilde{\Gamma}_{n+1,m-1}\hat{\tilde{E}}_{n+1,m}(\tilde{\Gamma}^1_{n+1,m-1})^T + \tilde{\mathcal{K}}_{n+1,m-1}(\tilde{\mathcal{K}}^1_{n+1,m-1})^T. \tag{3.2}$$

Proof. From the definition of $\hat{E}_{n,m}$,

$$\hat{E}_{n,m} = \langle z\Phi_{n-1,m}, \overleftarrow{\Phi}^T_{n-1,m}\rangle \tag{3.3}$$

eliminate $\Phi_{n-1,m}$ using equation (2.13) to obtain

$$\begin{aligned}\hat{E}_{n,m} &= \Gamma_{n-1,m+1}\langle z\Phi_{n-1,m+1}, \overleftarrow{\Phi}^T_{n-1,m}\rangle \\ &\quad + K_{n-1,m+1}\langle z\tilde{\Phi}_{n-2,m+1}, \overleftarrow{\Phi}^T_{n-1,m}\rangle.\end{aligned}$$

With the use of the reverse of (2.14) the first integral on the right hand side of the above equation can be rewritten as

$$\begin{aligned}\langle z\Phi_{n-1,m+1}, \overleftarrow{\Phi}^T_{n-1,m}\rangle &= \langle z\Phi_{n-1,m+1}, \overleftarrow{\Phi}^T_{n-1,m+1}\rangle(\Gamma^1_{n-1,m})^T \\ &\quad + \langle\Phi_{n-1,m+1}, \Phi_{n-2,m+1}\rangle(\mathcal{K}^1_{n-1,m+1})^T \\ &= \hat{E}_{n,m+1}(\Gamma^1_{n-1.m+1})^T.\end{aligned}$$

Equation (2.17) and the orthogonality of $\Phi_{n-1,m+1}$ to $\tilde{\Phi}_{n-n,m+1}$ has been used to obtain the last equality. The result now follows by taking the transpose of equation (2.21). Equation (3.2) follows in a similar manner using the tilde analog of the above equations. ☐

Lemma 3.2.

$$\Gamma_{n-1,m}\hat{E}_{n,m}I^1_{n-1,m} = A_{n,m-1}\mathcal{K}_{n,m} - \mathcal{K}_{n-1,m}\tilde{\Gamma}^1_{n-1,m} \tag{3.4}$$

and

$$\tilde{\Gamma}_{n,m-1}\hat{\tilde{E}}_{n,m}\tilde{I}^1_{n,m-1} = \tilde{A}_{n-1,m}\tilde{\mathcal{K}}_{n,m} - \tilde{\mathcal{K}}_{n,m-1}\Gamma^1_{n,m-1} \tag{3.5}$$

Proof. To obtain (3.4) note that equations (2.17) and the reverse transpose of (˜2.16) give,

$$\hat{E}_{n,m}I^1_{n-1,m} = \langle z\Phi_{n-1,m}, \tilde{\Phi}_{n-1,m}\rangle.$$

Multiplying the above equation on the left by $\Gamma_{n-1,m}$ then using the reverse transpose of equation (2.13) yields,

$$\Gamma_{n-1,m}\hat{E}_{n,m}I^1_{n-1,m} = \langle z\Phi_{n-1,m-1}, \tilde{\Phi}_{n-1,m}\rangle - \mathcal{K}_{n-1,m}\langle z\tilde{\Phi}_{n-2,m}, \tilde{\Phi}_{n-1,m}\rangle.$$

The second integral in the above equation evaluates to $\tilde{\Gamma}^1_{n-1,m}$. Substitution of equation (2.11) in the first integral to eliminate $z\Phi_{n-1,m-1}$ then using equation (2.19) yields equation (3.4). The argument for equation (3.5) follows in an analogous manner using the tilde analog of the above equations. □

Finally

Lemma 3.3.

$$I^\dagger_{n-1,m}\hat{E}_{n,m}(\Gamma^1_{n-1,m})^T = (\mathcal{K}^1_{n,m})^T A^T_{n,m-1} - \tilde{\Gamma}^\dagger_{n-1,m}(\mathcal{K}^1_{n-1,m})^T \tag{3.6}$$

and

$$\tilde{I}^\dagger_{n,m-1}\hat{\tilde{E}}_{n,m}(\tilde{\Gamma}^1_{n,m-1})^T = (\tilde{\mathcal{K}}^1_{n,m})^T \tilde{A}^T_{n-1,m} - \Gamma^\dagger_{n,m-1}(\tilde{\mathcal{K}}^1_{n,m-1})^T \tag{3.7}$$

Proof. To obtain (3.4) use equations (2.17) and (˜2.15) to find

$$I^\dagger_{n-1,m}\hat{E}_{n,m} = \langle z\tilde{\Phi}_{n-1,m}, \overleftarrow{\Phi}^T_{n-1,m}\rangle.$$

Multiplying the above equation on the left by the transpose of $\Gamma^1_{n-1,m}$ then using the reverse transpose of equation (2.14) yields

$$\begin{aligned} I^\dagger_{n-1,m}\hat{E}_{n,m}(\Gamma^1_{n-1,m})^T = \langle z\tilde{\Phi}_{n-1,m}, \overleftarrow{\Phi}_{n-1,m-1}\rangle \\ - \langle\tilde{\Phi}_{n-1,m}, \tilde{\Phi}_{n-2,m}\rangle(\mathcal{K}^1_{n-1,m})^T. \end{aligned}$$

The second integral in the above equation evaluates to $\tilde{\Gamma}^\dagger_{n-1,m}$. Substitution of the reverse transpose of equation (2.11) in the first integral then using equation (˜2.21) yields equation (3.6). As above equation (3.7) follows a similar argument using the tilde analogs of the above equations. □

With these recurrences we can prove a strengthening of Lemma 7.5 in [8]

Lemma 3.4. *If $\hat{E}_{i,j} = 0$, then the first column of $\mathcal{K}^1_{i,j}$ is equal to zero, in particular $u_{i,j} = 0$. If $\hat{E}_{i,j}$ and $\mathcal{K}_{i-1,j}(\mathcal{K}^1_{i-1,j})^T$ are zero, then so is $\hat{E}_{i,j-1}$. If $\mathcal{K}_{i,j}$, $\hat{E}_{i,j-1}$, and $u_{i,j}$ are zero, then $\hat{E}_{i,j} = 0$. Likewise if $\hat{\tilde{E}}_{i,j} = 0$, then the first row of $K^1_{i,j}$ is equal to zero. If $\hat{\tilde{E}}_{i,j}$, and $\mathcal{K}^\dagger_{i,j-1}\mathcal{K}^1_{i,j-1}$ are zero, then so is $\hat{\tilde{E}}_{i,j-1}$. If $\mathcal{K}_{i,j}$, $\hat{\tilde{E}}_{i-1,j}$, and $u_{i,j}$ are zero, then $\hat{\tilde{E}}_{i,j} = 0$.*

Proof. If $\hat{E}_{i,j} = 0$, then (3.6) and the triangular structure of $\tilde{\Gamma}_{i-1,j}$ show that the first column of $\mathcal{K}^1_{i,j}$ is zero. If $\hat{E}_{i,j}$ and $\mathcal{K}_{i-1,j}(\mathcal{K}^1_{i-1,j})^T$ are equal to zero, then (3.1) shows that $\hat{E}_{i,j-1} = 0$. Equations (3.50) and (3.51) in [8] are

$$\Gamma_{n-1,m}\hat{E}_{n,m} = A_{n,m-1}\mathcal{K}_{n,m}(I^1_{n-1,m})^\dagger + \hat{E}_{n,m-1}\bar{\Gamma}^1_{n-1,m}, \tag{3.8}$$

$$\hat{E}_{n,m}(\Gamma^1_{n-1,m})^T = I_{n-1,m}(\mathcal{K}^1_{n,m})^T A^T_{n,m-1} + \Gamma^\dagger_{n-1,m}\hat{E}_{n,m-1}. \tag{3.9}$$

Thus if $\mathcal{K}_{i,j}$ and $\hat{\tilde{E}}_{i-1,j}$ are equal to zero (3.8) and the fact that $\hat{E}_{i,j}$ is symmetric shows that all its entries are equal to zero except for $[\hat{E}_{i,j}]_{(1,1)}$. Equation (3.9) and the assumption that $u_{i,j} = 0$ give that this entry is equal to zero also. The remaining statements follow in an analogous fashion using the tilde analogs of the above equations.

□

For the next lemma we recast equations (2.5) and (2.7) as,

$$\Phi_{n,m}(z,w) = \sum_{i=0}^{m} L^i_{n,m} w^i [z^n, \dots, 1]^T \tag{3.10}$$

and

$$\tilde{\Phi}_{n,m}(z,w) = \sum_{i=0}^{n} \tilde{L}^i_{n,m} z^i [w^m, \dots, 1]^T \tag{3.11}$$

From this we have,

Lemma 3.5. *For $n, m \geq 0$*

$$I_{n,m} = L^m_{n,m} (\tilde{\Phi}^n_{m,m})^{-1} \tag{3.12}$$

and

$$I^1_{n,m} = \bar{L}^0_{n,m} J_n (\tilde{\Phi}^n_{m,m})^{-1}, \tag{3.13}$$

where J_n is the $(n+1) \times (n+1)$ matrix with ones on the reverse-diagonal and zeros everywhere else.

Proof. The first formula follows from equation (3.10). The second equation can be seen from the computation,

$$\overleftarrow{\Phi}_{n,m}(z,w)^T = \sum_{i=0}^{m} \bar{L}^i_{n,m} w^{m-i} J_n [z^n, \dots, 1]^T.$$

□

The above results give formulas for the parameters. From equations (2.35), (2.37), and (3.12) we find

$$(e^j_j)^T (\Phi^{j-1}_{i,i})^{-1} \mathcal{K}_{i,j} ((\tilde{\Phi}^{i-1}_{j,j})^\dagger)^{-1} e^i_i = -(e^{j+1}_{j+1})^T (\Phi^j_{i,i})^{-1} L^j_{i,j} U^T_i ((\tilde{\Phi}^{i-1}_{j,j})^\dagger \tilde{\Phi}^{i-1}_{j,j})^{-1} e^i_i, \tag{3.14}$$

where we have used the fact the $U^T_j e^j_j = e^{j+1}_{j+1}$ in the last equation. Likewise equations (2.36), (2.38), and (3.13) show,

$$(\mathcal{K}^1_{i,j})_{1,1} = -(\Phi^{j-1}_{i,i} U_i (\Phi^j_{i,i})^{-1} \bar{L}^0_{i,j} J_i (U^1_i)^T ((\tilde{\Phi}^{i-1}_{j,j})^\dagger \tilde{\Phi}^{i-1}_{j,j})^{-1})_{1,1}. \tag{3.15}$$

4. Construction of the Parameters

Using the results above we are now able to compute the remaining parameters from those given in the rectangle $0 \leq i \leq n,\ 0 \leq j \leq m$. We first show that all the polynomials $\Phi_{i,j}$ and $\tilde{\Phi}_{i,j}$ can be computed for $i > n$ and $0 \leq j \leq m$. To see this suppose that we are given $\Phi_{n,j}$ and $\tilde{\Phi}_{n,j}$ for $0 \leq j \leq m$ and also conditions (a), (b), and (c) of Theorem 2.4 are satisfied. From their defining properties we see that $\phi^m_{n,m} = \tilde{\phi}^n_{n,m}$. Lemma 3.4 shows that (a) and (b) imply that $\hat{E}_{i,m} = 0$ for $i > n$ so that from equation (2.11) $\Phi_{i,m} = z^{i-n}\Phi_{n,m}$ and because $\hat{E}_{i,m-1} = 0,\ i > n,\ \Phi_{i,m-1} = z^{i-n}\Phi_{n,m-1}$. If $\mathcal{K}_{i,m} = 0$ it follows from the triangularity of $\Gamma_{i,m}$ and equation (2.28) that $\Gamma_{i,m} = U_m$. Likewise $\tilde{\Gamma}_{i,m} = U_i$. This implies through equations (˜2.13) and (˜2.23) that

$$\tilde{\Phi}_{i,m} = \begin{bmatrix} z^{i-n}\phi^m_{n,m} \\ \tilde{\Phi}_{i-1,m} \end{bmatrix}$$

for $i \geq n$ which gives all of $\tilde{\Phi}_{i,m}$ for $i > n$. Set $w = 0$ in (˜2.12) then utilize equation (2.40) and the fact that $\overleftarrow{\tilde{\Phi}}{}^n_i(0)$ is invertible to obtain

$$\tilde{A}^\dagger_{i,m}\hat{\tilde{E}}_{i,m}(\tilde{A}^T_{i,m})^{-1} = -\tilde{\Phi}^i_m(0)J_i(\overleftarrow{\tilde{\Phi}}{}^i_m(0)^{-1})^T \equiv B^i_m. \tag{4.1}$$

To find $\tilde{A}_{i,m}$ use the orthogonality properties of $\tilde{\Phi}_{i,m}$ and $\overleftarrow{\tilde{\Phi}}_{i,m}$ in equation (˜2.12) to find,

$$I - B^i_m(B^i_m)^\dagger = \tilde{A}^\dagger_{i,m}\tilde{A}_{i,m}.$$

Since $\tilde{A}_{i,m}$ is upper triangular with positive diagonal entries it may be computed using the lower Cholesky factorization of the left hand side of the above equation. Using this in (˜2.12) allows us to compute $\tilde{\Phi}_{i,m-1}$ for $i > n$. In an analogous fashion $\tilde{\Phi}_{i,j}$ may be computed for $i > n$ and $0 \leq j \leq m-1$. Now $\tilde{\Gamma}_{i,j}$ and $\tilde{\Gamma}^1_{i,j}$, $i > n$, $0 < j \leq m-1$ may be computed from from equations (˜2.35) and (˜2.36) respectively. With $i = m-1$ we find from (3.1) since $\hat{E}_{n+1,m-1} = 0$ that,

$$\hat{E}_{n+1,m-2} = \mathcal{K}_{n,m-1}(\mathcal{K}^1_{n,m-1})^T$$

which gives $\hat{E}_{n+1,m-2}$ because by assumption $\mathcal{K}_{n+1,m-1}$ and $\mathcal{K}^1_{n+1,m-1}$ are known. Since $A_{n+1,m-2}$ may be computed from $\hat{E}_{n+1,m-2}$ using the upper Cholesky factorization of (2.27) we obtain $\Phi_{n+1,m-1}$ from (2.11). By induction we see that the above argument gives $\hat{E}_{n+1,i}$, $i = 0,\ldots,m-3$ from which $A_{n+1,i}$ may be computed and then $\Phi_{n+1,i}$. Using equations (3.12), (3.13), (2.35)–(2.38) allows us to compute $\Gamma_{n+1,i}, \Gamma^1_{n+1,i}$, $I_{n+1,i}$, $I^1_{n+1,i}$, $\mathcal{K}_{n+1,i}$, and $\mathcal{K}^1_{n+1,i}$ for $0 < i \leq m-1$. With equation (3.1) and the coefficients just computed we repeat the above argument for level $(n+2,i)$ and by induction (i,j), $i > n$, $0 \leq j \leq m-1$.

We summarize this with

Lemma 4.1. *Given (a), (b), and (c) of Theorem* 2.4 *as well as* $\Phi_{n,j}$ *and* $\tilde{\Phi}_{n,j}$ *for* $0 \leq j \leq m$ *then* $\Phi_{i,j}$ *and* $\tilde{\Phi}_{i,j}$ *for* $i > n,\ 0 \leq j \leq m$ *can be computed recursively. If* $\Phi_{i,m}$ *and* $\tilde{\Phi}_{i,m}$ *are given then* $\Phi_{i,j}$ *and* $\tilde{\Phi}_{i,j}$ *for* $0 \leq i \leq n,\ j > m$ *can be computed recursively.*

We now use the formulas (3.14) and (3.15) which give $u_{-i,j}$ for $i > n$ and $1 \leq j < m$ and $u_{i,j}$ for $i > n$ and $1 \leq j < m-1$. For $j = 0$ we have from [8] that

$$u_{i,0} = -\frac{\Phi^0_{i,0}}{\Phi^0_{i,i}}, \tag{4.2}$$

which gives the parameters in the strip $i > n,\ 0 \leq j \leq m-1$. To compute the parameters in the strip $j > m,\ 0 \leq i \leq n-1$ equations (2.25) and (2.26) show that we need only interchange i with j and the matrices associated with the lexicographical ordering with those associated with the reverse lexicographical ordering in equations (3.14), (3.15), and (4.2). This leads to,

Theorem 4.2. *Suppose* σ *is a positive Borel measure supported on the bi-circle with parameters* $u_{i,j}$. *If*

(a) $u_{i,-j}, 1 \leq i,j \leq n$ *and* $u_{i,j}, 0 \leq i,j \leq n$ *give* $\mathcal{K}_{n,m} = 0$*;*

(b) $u_{i,j} = 0$ *for* $i = n-1, j > m,\ i > n, j = m-1,\ |i| > n, j = m,\ |i| \geq n, j > m$*;*

(c) *for* $i > n,\ 0 \leq j \leq m-2,\ u_{i,j}$ *are equal to the left hand sides of (3.14), (3.15), or (4.2)computed using the above algorithm;*

(d) *for $j > m$, $0 \leq i \leq n-2$, $u_{i,j}$ are equal to the left hand sides of (˜3.14), (˜3.15), of (˜4.2) computed using the above algorithm,*

then σ is absolutely continuous with respect to Lebesgue measure with density $\frac{1}{|p_{n,m}|^2}$ where $p_{n,m}$ is of degree (n,m) with $\overleftarrow{p}_{n,m}(z,w)$ stable. $p_{n,m}$ is unique up to multiplication by a complex number of modulus one.

Proof. If $\mathcal{K}_{n,m} = 0$ then from Theorem (7.3) of [8] or Theorem (10.1) of [11] there exits a polynomial $p_{n,m}$ of degree (n,m) with $\overleftarrow{p}_{n,m}$ stable such that the measure $d\rho = \frac{d\theta d\phi}{4\pi^2|p_{n,m}|^2}$ has parameters given in (a). By Theorem (2.4) and the algorithm above we that this measure has the same paramters as in (b) and (c). Thus by Theorem (2.3) $d\sigma = \frac{d\theta d\phi}{4\pi^2|p_{n,m}|^2}$.

□

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

E-mail address: geronimo@math.gatech.edu

4214 Swire Ave., Apt 8, Baton Rouge, LA 70808, USA

E-mail address: pbenge2@tigers.lsu.edu